Example:- Let $1 \leq p < \infty$. Let l^p be the set of all sequences $x = \{x_i\}$ such that $\sum_{i=1}^{\infty} |x_i|^p < \infty$. then l^p is a Banach space w.r.t. addition scalar multiplication and norm defined as follows

for any $x = \{x_i\}$, $y = \{y_i\}$ we define $x + y = \{x_i + y_i\}$ for any scalar λ, we define $\lambda x = \{\lambda x\}$ and $\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$.

Verification:- l^p is clearly a linear space w.r.t. the above definition of addition and scalar multiplication. Clearly, $\|x\| \geq 0$. $\|x\| = 0$ iff $\left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} = 0$ iff $\sum_{i=1}^{\infty} |x_i|^p = 0$ iff $|x_i|^p = 0$ for $i = 1, 2, ----$ iff $x_i = 0$ for $i = 1, 2, ----$ iff $x = 0$

Also, $\|\lambda x\| = \left(\sum_{i=1}^{\infty} |\lambda x_i|^p\right)^{1/p}$

$$= |\lambda| \cdot \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} = |\lambda| \cdot \|x\|$$

In order to verify the fourth Condition, we need Minkowski's inequality

$$\left(\sum_{i=1}^{\infty} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{1/p}.$$

From this, we have $\|x+y\| \leq \|x\| + \|y\|$
$\therefore l^p$ is a normed linear space. The metric d generated by the norm is given by

$$d(x,y) = \|x-y\| = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{1/p}.$$

We are required to show that (l^p, d) is a Complete metric space.

Let $\{x(m)\}$ be any Cauchy sequence in l^p. Where $x(m) = \{x_i(m)\}$ then, given $\epsilon > 0$, there exists a +ve integer n_0 such that $d(x(m), x(n)) = \left[\sum_{i=1}^{\infty} |x_i(m) - x_i(n)|\right]^{1/p} < \epsilon$

for all $m, n \geq n_0$ i.e. $\sum_{i=1}^{\infty} |x_i(m) - x_i(n)|^p < \epsilon^p$

~~for all m, n~~

i.e $\sum_{i=1}^{\infty} |x_i(m) - x_i(n)|^p < \epsilon^p$ for all $m, n \geq n_0$ — (1)

$\therefore$ For each fixed i,

$$|x_i(m) - x_i(n)|^p < \sum_{i=1}^{\infty} |x_i(m) - x_i(n)|^p < \epsilon^p$$

for all $m, n \geq n_0$.

i.e. $|x_i(m) - x_i(n)| < \epsilon$ for all $m, n \geq n_0$ — (2)

$\therefore$ For each fixed i, $\{x_i(n)\}$ is a Cauchy sequence of real numbers. So, there exists a real no. x_i such that, $x_i(n) \to x_i$

Let $x = \{x_i\}$. then letting $m \to \infty$ in (1),

$$\sum_{i=1}^{\infty} |x_i - x_i(n)|^p < \epsilon^p \text{ for all } n \geq n_0.$$

i.e. $\left[\sum_{i=1}^{\infty} |x_i - x_i(n)|^p\right]^{1/p} < \epsilon$ for all $n \geq n_0$.

$\therefore x \to x^{(n)} \in l^p$. Since, l^p is linear space and $x(n) \in l^p$, $\therefore x \in l^p$.

Now, $d(x, x(m)) = \left[\sum_{i=1}^{\infty} |x_i - x_i(m)|^p\right]^{1/p} < \epsilon$ for all $m \geq n_0$.

$\therefore x(m) \to x$ as $m \to \infty$ i.e. l^p is complete metric space. So, it is a Banach space.

Example:- Let $C[a, b]$ be the space of all continuous real functions defined on $[a, b]$ for any $f, g \in C[a, b]$ and $\lambda \in \mathbb{R}$ we define

$$(f + g)(x) = f(x) + g(x) \text{ for all } x \in [a, b]$$

$(\lambda \cdot f)(x) = \lambda \cdot f(x)$ for all $x \in [a,b]$
and $\|f\| = \sup_{x\in[a,b]} |f(x)|$

Then, $C[a,b]$ is a Banach space.

<u>Verification:</u>- Clearly, $C[a,b]$ is a real linear space w.r.t. the above definition of addition and scalar multiplication.
Also, $\|f\| \geq 0$. Also, $\|f\| = 0$ iff $\sup_{x\in[a,b]} |f(x)| = 0$

iff $|f(x)| = 0$ for all $x \in [a,b]$ iff $f = 0$
Again, $\|\lambda f\| = \sup_{x\in[a,b]} |(\lambda f)(x)|$.

$= |\lambda| \cdot \sup_{x\in[a,b]} |f(x)| = |\lambda| \cdot \|f\|$.

Finally, $|f(x) + g(x)| \leq |f(x)| + |g(x)|$ for all $x \in [a,b]$.
$\therefore |f(x) + g(x)| \leq \sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |g(x)|$

for all $x \in [a,b]$
i.e. $|f(x) + g(x)| \leq \|f\| + \|g\|$ for all $x \in [a,b]$.
$\therefore \sup |f(x) + g(x)| \leq \|f| + |g\|$ for all $x \in [a,b]$
$\therefore \sup_{x\in[a,b]} |f(x) + g(x)| = \|f\| + \|g\|$

i.e. $\|f + g\| \leq \|f\| + \|g\|$
$\therefore C[a,b]$ is a real normed linear space.

The metric d generated by the norm

is given by $d(f, g) = \|f - g\|$
$$= \sup_{x \in [a, b]} |f(x) - g(x)|$$

It remains to inequality that $(C[a, b], d)$ is a complete.

Let $\{f_n\}$ be any Cauchy sequence in $C[a, b]$ then,

$$d(f_n, f_m) = \sup_{x \in [a, b]} |f_n(x) - f_m(x)| < \epsilon \text{ for}$$

all $m, n \geq n_0$.

i.e. $|f_n(x) - f_m(x)| <$ for all $m, n \geq n_0$ and all $x \in [a, b]$ ——— (1)

$\therefore$ For each $x \in [a, b]$ $\{f_n(x)\}$ is a Cauchy sequence of real numbers and so converges to some real number $f(x)$ (suppose). We now define a function, f on $[a, b]: x \to f(x)$

Now, letting, $m \to \infty$ in (1) we get
$|f_n(x) - f(x)| < \epsilon$ for all $n \geq n_0$

$\therefore \{f_n\}$ converges uniformly to f. So f is continuous i.e. $f \in C[a, b]$.

Hence, $C[a, b]$ is a complete metric space. i.e. a Banach space.

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Example:- An example of a normed linear space (n.l.s.) which is not a Banach space.

Let E be the linear space of all polynomial defined over $[0, 1]$ with ~~real~~ real

Co-efficients. For any Polynomial $p \in E$, we define.

$$\|p\| = \sup_{t\in[0,1]} |p(t)|.$$

Clearly, $\|p\| \geq 0$. Also $\|p\| = 0$ iff $p = 0$

Again, $\|\alpha p\| = \sup_{t\in[0,1]} |\alpha p(t)|.$

$$= \sup_{t\in[0,1]} |\alpha . p(t)| = |\alpha| . \sup_{t\in[0,1]} |p(t)|.$$

$$= |\alpha| . \|p\|.$$

Now, for any $p, q \in E$,

$$\|p+q\| = \sup_{t\in[0,1]} |(p+q)(t)|$$

$$= \sup_{t\in[0,1]} |p(t) + q(t)|$$

$$\leq \sup_{t\in[0,1]} [|p(t)| + |q(t)|]$$

$$= \sup_{t\in[0,1]} |p(t)| + \sup_{t\in[0,1]} |q(t)|$$

$$= \|p\| + \|q\|$$

So, $\|p+q\| \leq \|p\| + \|q\|$.

$\therefore$ E is a normed linear space.

The metric d generated by the norm on E is given by,

$$d(p, q) = \|p-q\| = \sup_{t\in[0,1]} |p(t) - q(t)|.$$

We shall show that (E, d) is not a complex variable space.

We Consider a sequence $\{p_n\}$ of Polynomials in E, where $p_n(t) = 1 + t + \frac{t^2}{\lfloor 2} + \cdots + \frac{t^n}{\lfloor n}$.

Let $\epsilon > 0$ be given, then $d(p_n, p_m)$.

$$= \| p_n - p_m \|.$$

$$= \sup_{t \in [0,1]} | p_n(t) - p_m(t) |$$

$$= \sup_{t \in [0,1]} \left| \frac{t^{m+1}}{\lfloor m+1} + \cdots + \frac{t^n}{\lfloor n} \right|$$

$$\leq \frac{1}{\lfloor m+1} + \cdots + \frac{1}{\lfloor n} \to 0$$

as $m, n \to \infty$.

So, $\{p_n\}$ is a Cauchy sequence in E.

But $\lim_{n \to \infty} p_n(t)\ 1 + t + \frac{t^2}{\lfloor 2} + \cdots$

$= e^t$.

So, $\{p_n\}(t)\}$ Converges to e^t, which is not a Polynomial in E. So $e^t \notin E$. Since there exist a Cauchy sequence in E, which does not converges in E,

So, E is not a Banach space.

Example:- An example of metric space which is not a normed linear space.

Let S be the linear space of all numerical sequences, For $x = \{x_i\}$ & $y = \{y_i\}$

in S, we define $x+y=\{x_i+y_i\}$ & $\lambda x=\{\lambda x_i\}$.
We consider a metric ρ on S defined by,

$$\rho(x,y)=\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|x_i-y_i|}{1+|x_i-y_i|}$$

Then, (S,ρ) is a metric space.
We shall show that it is not possible to define a normed on S which generate the metric ρ. For let there be a norm on S and let d be the metric generated by this norm.
It is sufficient to show that d is not equivalent to ρ.

For this we consider a sequence $e^{(i)}=\{0,0,\ldots,1,0,0\ldots\}\in S$.

We define $x^{(i)}=\dfrac{e^i}{\|e^i\|}$, $i=1,2,3,\ldots$

Then, $\rho(x^{(i)},0)\leq\frac{1}{2^i}\to 0$ as $i\to\infty$

So, $x^i \xrightarrow{\rho} 0$

But $d(x^{(i)},0)=\|x^{(i)}-0\|=\|x^{(i)}\|=1$

So, $x^i \not\xrightarrow{d} 0$

Then, d is not equivalent to ρ.
So, S is not a normed linear space.